Dimensional reduction in supersymmetric field theories

Oleg V. Zaboronsky *
Department of Mathematics, University of California at Davis,
Davis, CA 95616; e-mail: zaboron@math.ucdavis.edu

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Abstract

A class of quantum field theories invariant with respect to the action of an odd vector field Q on a source supermanifold Σ is considered. We suppose that Q satisfies the conditions of one of localization theorems (see e. g. [9]). The Q-invariant sector of a field theory from the class above is shown then to be equivalent to the quantum field theory defined on the zero locus of the vector field Q.

1 Introduction

The aim of the present paper is to connect the phenomenon of dimensional reduction in supersymmetric field theories with localization of certain integrals over supermanifolds. Let us start with the explanation of the meaning we assign to the terms "dimensional reduction" and "localization" (each of these terms is used in the literature in quite a few different contexts). By dimensional reduction we will understand a fact of an *exact* equivalence between a quantum field theory and another quantum field theory defined on the submanifold of the source manifold of the original theory. An example of such phenomenon is provided by the celebrated Parisi-Sourlas model [4] which also served as a main motivation for the present work.

Localization of an integral over a (super)manifold Σ to a subset $R \subset \Sigma$ means more or less that this integral is independent of the values of the integrand on the complement to the arbitrary neighborhood of R in Σ. In what follows we will be using the notion of localization in the even more restricted sense. It is well-known that localization is usually related to the presence of some odd symmetry of the problem. So let Q be an odd vector field on Σ. This means

*On leave from the Institute of Theoretical and Experimental Physics, Moscow, Russia.

that Q is a parity-reversing derivation on Z_2-graded algebra of functions on Σ. We say that Q satisfies the conditions of some localization theorem if for any Q-invariant function f on Σ

$$\int_{\Sigma} dV \cdot f = \int_{R_Q} dv_Q f \mid_{R_Q}, \tag{1}$$

where dV is a fixed volume element on Σ; zero locus of Q is supposed to be a submanifold of Σ and is denoted by R_Q; dv_Q stands for the volume element on R_Q depending on dV, Q, but not f.

An exposition of different localization techniques in the context of quantum field theory can be found in [10]. In [9] we studied localization in the framework of supergeometry. We managed to prove a general localization theorem which includes essentially all previously known localization statements as its particular cases. The main result of [9] can be formulated as follows. Let Q be an odd vector field on Σ which preserves a volume element dV on Σ . Suppose that $Q^2 = \frac{1}{2}\{Q, Q\}$ belongs to a Lie algebra of a compact subgroup of the group of diffeomorphisms of Σ. Then under some additional conditions of non-degeneracy of Q the integrals of Q-invariant functions over Σ localize to the zero locus R_Q of the vector field Q. In other words (1) holds with dv_Q determined by dV and the matrix of the first derivatives of the vector field Q at R_Q.

To conclude the Introduction let us formulate and prove another localization theorem which will be useful in the analysis of dimensional reduction of Parisi-Sourlas-type models.

The explanation of basic notions of supergeometry which will be used below can be found in [8].

Theorem. *Let Σ be a compact supermanifold equipped with an even metric g. Suppose Q is an odd vector field on Σ preserving the metric, i. e. $L_Q g = 0$. Suppose that vector field Q^2 is non-degenerate in the vicinity of its zero locus R_{Q^2}. Suppose also that odd and even codimensions of R_{Q^2} in Σ coincide. Then for any Q-invariant function f on Σ*

$$\int_{\Sigma} dV f = \int_{R_{Q^2}} dv_Q f \mid_{R_{Q^2}}, \tag{2}$$

where dV is a volume element on Σ corresponding to the metric g and dv_Q is a volume element on R_{Q^2} determined completely by g and Q. [1].

Proof. Let $\{z^\alpha\}$ be a set of local coordinates on Σ. The parity of the α'th coordinate will be denoted by ϵ_α. In these coordinates $Q = Q^\alpha(z)\frac{\partial}{\partial z^\alpha}, Q^2 = (Q^2)^\alpha(z)\frac{\partial}{\partial z^\alpha}$, where $(Q^2)^\alpha = Q(Q^\alpha(z))$. We will write the metric in the form $g = g_{\alpha\ \beta}(z)\delta z^\alpha \delta z^\beta$. Consider now an odd function σ on Σ defined in the local

[1] Initially this theorem was formulated and proved for the linear superspaces. Its present form benefits from the collaboration with A. S. Schwarz

coordinates by the following expression:

$$\sigma(z) = \frac{1}{2}\sum_{\alpha,\beta}(-1)^{\epsilon_\alpha+\epsilon_\beta} g_{\alpha\ \beta} Q^\alpha(z)(Q^2)^\beta(z) \tag{3}$$

It is easy to verify that the right hand side of (3) does not depend on the choice of local coordinates, so that σ is indeed a function on Σ. A direct calculation shows that σ is Q^2-invariant, i. e. $Q^2\sigma = 0$. Also one finds that $Q\sigma(z) = g_{\alpha\ \beta}(Q^2)^\alpha(z)(Q^2)^\beta(z) \equiv < Q^2(z), Q^2(z) >$, where $<,>$ denotes the pairing in the fibers of the tangent bundle over Σ induced by the metric g.

Another computation shows that R_{Q^2} is a subset of the critical set of $Q\sigma$, i. e. $\nabla Q\sigma \mid_{R_{Q^2}} = 0$. It follows from non-degeneracy of Q^2 in the vicinity of R_{Q^2} that R_{Q^2} is a non-degenerate critical set. The last means that the Hessian of $Q\sigma$ has the maximal rank at each point of R_{Q^2}.

Our aim is to compute $\int_\Sigma dV f$, where $Qf = 0$. The fact that the metric g is Q-invariant implies that $div_{dV} Q = 0$, which means that the volume element on Σ constructed using the Q-invariant metric is also Q-invariant. Thus it is easy to see that the following is true:

$$\frac{\partial}{\partial\lambda}\int_\Sigma dV \cdot f e^{-\lambda Q\sigma} = -\int_\Sigma dV Q(f e^{-\lambda Q\sigma}) = 0 \tag{4}$$

Let $\{U_m\}_{m\in I}$ be a finite atlas of Σ, $\{h_m\}_{m\in I}$ - a partition of unity on Σ subordinate to this atlas. Suppose also that the atlas is chosen to satisfy the following two condition:

(i) if $\overline{U_k}\bigcap R_{Q^2} \neq \emptyset, k \in I$, then $U_k \bigcap R_{Q^2} \neq \emptyset$;

(ii) if $U_k \bigcap R_{Q^2} \neq \emptyset$ then the critical set of $Q\sigma \mid_{U_k}$ is just $U_k \bigcap R_{Q^2}$.

Using (4) one can rewrite an expression for the integral of f over Σ as follows:

$$\int_\Sigma dV\ f = \lim_{\lambda\to\infty}\int_\Sigma dV\ f e^{-\lambda <Q^2,Q^2>} =$$
$$\sum_{m\in I}\lim_{\lambda\to\infty}\int_{U_m} dV h_m f e^{-\lambda <Q^2,Q^2>} =$$

Let $m(Q^2)$ denotes a number part of the vector field Q^2 , $R_{m(Q^2)} \subset \Sigma$ denotes zero locus of $m(Q^2)$. Let us choose $k \in I\ :\ \overline{U_k}\bigcap R_{Q^2} = \emptyset$. Then $\overline{U_k}\bigcap R_{m(Q^2)} = \emptyset$. As a consequence of (i) the number part of $< Q^2, Q^2 >$ is positive at each point of U_k, so one can find such positive constants c_1 and c_2 that:

$$\mid \int_{U_k} dV h_k f e^{-\lambda <Q^2,Q^2>} \mid \leq c_1\lambda^n e^{-c_2\lambda} \to 0 \quad \text{as} \quad \lambda\to\infty$$

Thus we conclude that

$$\int_\Sigma dV\ f = \sum_{\{k\in I \mid U_k \bigcap R_{Q^2}\neq\emptyset\}} \lim_{\lambda\to\infty}\int_{U_k} dV h_k f e^{-\lambda <Q^2,Q^2>} \tag{5}$$

The integrals in the right hand side of (5) can be calculated using the Laplace method adapted to include integrals over superspaces (see e. g. [5]). Under the condition that odd codimension of $R_{Q^2} \subset \Sigma$ is equal to its even codimension we obtain that

$$\lim_{\lambda\to\infty} \int_{U_k} dV h_k f e^{-\lambda<Q^2,Q^2>} = \int_{U_k \bigcap R_{Q^2}} dv_Q (h_k f) \mid_{U_k \bigcap R_{Q^2}}, \tag{6}$$

where dv_Q is the volume element on $U_k \bigcap R_{Q^2}$ defined as a partition function of degenerate functional $Q\sigma \mid_{U_k}$ (see [5], lemma 2). By (5) $Q\sigma$ depends on Q and g only, so does dv_Q. It remains to notice that the set $\{h_k \mid_{R_{Q^2}}\}_{\{k\in I|U_k \bigcap R_{Q^2}\neq\emptyset\}}$ provides one with the partition of unity on R_{Q^2}. Therefore substituting (6) into (5) and using the definition of the integral over a (super)manifold we see that

$$\int_\Sigma dV \cdot f = \int_{R_{Q^2}} dv_Q \cdot f \mid_{R_{Q^2}}$$

The Theorem is proved.

Corollary. *If $R_{Q^2} = R_Q$ the Theorem implies the localization of corresponding integrals over Σ in the sense of the definition (1).*

Note that the statement of the Theorem above can be formally justified in the case when Σ is an infinite-dimensional manifold. For example Σ can be realized as a space of maps from a world sheet to a target manifold of some quantum field theory. This suggests that there are possible applications of the Theorem above which are different from the ones we consider below.

Finally let us remark that if $R_{Q^2} \neq R_Q$ one can still prove the localization of the integrals under consideration to R_Q. The proof will consist of two steps: first one repeats the arguments above to prove the localization to R_{Q^2}. Then one notes that the vector field Q generates a nilpotent vector field on R_{Q^2} and $f \mid_{R_{Q^2}}$ is invariant with respect to this vector field. Corresponding integral is localized to R_Q (see e. g. [11], [7]).

2 Derivation of the main result.

Let Σ be a compact supermanifold. Suppose that Q is an odd vector field on Σ. We always assume that zero locus R_Q of the vector field Q is a submanifold of Σ and that Q is non-degenerate in the neighborhood of R_Q. Let dV be a fixed Q-invariant volume element on Σ. Assume that Q satisfies the localization conditions, i. e. (1) holds for any Q-invariant function f on Σ. Let M be another supermanifold. To avoid irrelevant technicalities we suppose that M is diffeomorphic to a linear superspace. Denote by E the (super)space of maps from Σ to M. Naturally an action of Q on Σ generates an infinitesimal diffeomorphism of the space of maps:

$$\Phi \to \Phi + \epsilon Q \Phi \ , \tag{7}$$

where $\Phi \in E$ and ϵ is an odd parameter. We will use the notation $\hat{Q}$ for the vector field on E corresponding to (7).

Next let us impose an additional condition on Q which will be crucial for further considerations. Namely we will assume that the following Cauchy problem has a solution:

$$Q\Phi = 0 \tag{8}$$

$$\Phi_{R_Q} = \Phi_0, \tag{9}$$

where Φ_0 is any map from R_Q to M. In other words we require that any map $R_Q \to M$ can be continued to the Q-invariant map $\Sigma \to M$. In all interesting cases the problem (8), (9) has a lot of solutions. We suppose that the space of solutions of (8) corresponding to a fixed initial condition (9) is contractible.

Consider now a quantum field theory defined on Σ. Let $\mathcal{L}(\Phi, \partial\Phi), \Phi \in E$ be a corresponding quantum Lagrangian. The word "quantum" means that having started from classical field theory we fixed gauge-like symmetries of the classical Lagrangian using some quantization procedure (BV for example, see [1]) and arrived at the expression for $\mathcal{L}(\Phi, \partial\Phi)$ where Φ is a map from Σ to the manifold M of both physical and auxiliary fields such as ghosts, antifields, etc. Therefore corresponding action functional is non-degenerate, i. e. the linear integral operator in $T_\Phi(E)$ with the kernel $\frac{\delta^2 S}{\delta\Phi^i(x)\Phi^j(y)}$ has no zero eigenvectors for any $\Phi \in E$. Here $x\ , y \in \Sigma$ and a choice of local coordinates in M is assumed.

The main condition imposed on the quantum field theory at hand is $\hat{Q}$-invariance. Namely we suppose that

$$\mathcal{L}(\Phi + \epsilon Q\Phi, \partial(\Phi + \epsilon Q\Phi)) = \mathcal{L}(\Phi, \partial\Phi) + \epsilon Q\mathcal{L}(\Phi, \partial\Phi) \tag{10}$$

The fact that the action $S = \int_\Sigma dV\mathcal{L}$ is $\hat{Q}$ is invariant follows then from (10) and the Q-invariance of the volume element dV.

There is a simple construction generating a lot of models satisfying (10). Let $h_n$be a multivector field of rank n on Σ. Introducing local coordinates $\{z^\alpha\}$ on Σ one can present h_n in the following form:

$$h_n = h_n^{\alpha_1\ \alpha_2\ldots\alpha_n}(z)\frac{\partial}{\partial z^{\alpha_1}} \otimes \frac{\partial}{\partial z^{\alpha_2}} \otimes \ldots \otimes \frac{\partial}{\partial z^{\alpha_n}}$$

Using a multivector field h_n and a map $\Phi : \Sigma \to M$ one can construct a map $h_{*\ n}$ from Σ to the n'th tensor power of the tangent bundle TM over M. Choosing local coordinates both in Σ and M one can present it as follows:

$$\begin{aligned} h_{*\ n}(z) = h_n^{\alpha_1\ \alpha_2\ldots\alpha_n}(p)\frac{\partial\Phi^{i_1}}{\partial z^{\alpha_1}}\frac{\partial\Phi^{i_2}}{\partial z^{\alpha_2}}\cdots\frac{\partial\Phi^{i_n}}{\partial z^{\alpha_2}} \\ \equiv h_n(\Phi^{i_1} \times \Phi^{i_2} \times \ldots \times \Phi^{i_n})(z) \end{aligned} \tag{11}$$

Suppose now that h_n is Q-invariant, i. e. $L_Q h_n = 0$, where L_Q is a Lie derivative with respect to the vector field Q. Then as it is easy to see

$$\epsilon Q(h_n(\Phi \times \ldots \times \Phi)) = \tag{12}$$

$$= h_n((\Phi + \epsilon Q\Phi) \times \ldots \times (\Phi + \epsilon Q\Phi)) - h_n(\Phi \times \ldots \times \Phi)$$

Suppose finally that the derivatives of Φ enter the Lagrangian only in the form of combinations (11), where h_n is a Q-invariant multivector field. Then in virtue of (12) the relation (10) is satisfied and corresponding action functional is $\hat{Q}$-invariant.

Let us illustrate the above considerations with the following example. Take g to be a Q-invariant metric on Σ. Then the following model is $\hat{Q}$-invariant:

$$S = \int_\Sigma dV(g^{\alpha\beta}\partial_\alpha\Phi^i\partial_\beta\Phi^j G_{ij}(\Phi) + V(\Phi)) \tag{13}$$

Here $g^{\alpha\beta}$ is a Q-invariant multivector field of rank 2 inverse to the metric tensor $g_{\alpha\beta}$ and G_{ij} is a metric tensor on M. Note that (10) constitutes a natural non-linear generalization of Parisi-Sourlas model [4].

Now we are able to formulate the main result of the paper. The Q-invariant (Schwinger) correlation functions of the theory described above have the following generating functional:

$$Z[J] = \int [D\Phi]_E e^{i\beta(S[\Phi] + \int_\Sigma dV J_i(p)\Phi^i(p))}, \tag{14}$$

where $\{J_i\}$ are Q-invariant functions on Σ playing a role of sources, $[D\Phi]_E$ is a formal measure on the space of maps E, β is a coupling constant.

By means of formal manipulations with functional integrals we are going to show that under the conditions on Q and $S[\Phi]$ the generating functional (14) can be rewritten as follows:

$$Z[J] = \int [D\Phi]_{E_Q} exp^{i\beta(S[\Phi|_{R_Q}] + \int_{R_Q} dv_Q J_i(p)\Phi^i(p)|_{R_Q})} \tag{15}$$

Here E_Q denotes the space of maps from R_Q to M, $[D\Phi]_{E_Q}$ is a measure on E_Q; the new action functional is

$$S[\Phi\mid_{R_Q}] = \int_{R_Q} dv_Q \mathcal{L}(\Phi\mid_{R_Q}, \partial'\Phi\mid_{R_Q}, 0) \tag{16}$$

where the new Lagrangian is obtained from the old one by restricting the fields to R_Q and setting the derivatives of the fields in the directions transversal to R_Q equal to 0. We also used the symbol ∂' to denote the derivatives along R_Q.

Eq. (15) states the equivalence between the Q-invariant sector of the initial theory and the theory determined by the action functional (16) defined on the submanifold of the initial source manifold Σ. According to the adopted terminology, dimensional reduction occurs.

Note that in the case when R_Q is zero-dimensional, the r. h. s of (15) reduces to a finite-dimensional integral, which means an exact solvability of the

Q-invariant sector of the theory we have started with. We also see that in the situation when Q happens to have no zeros at all the Q-invariant sector is trivial which yields a set of Ward identities for the correlation functions of the initial theory.

To demonstrate the equality between (14) and (15) let us consider first the subset $\mathcal{R}_Q$ of E consisting of Q-invariant maps from Σ to M. The space $\mathcal{R}_Q$ is foliated by means of the following equivalence relation: two Q-invariant maps $\Phi, \Phi' \in \mathcal{R}_Q$ belong to the same fibre of the foliation iff $\Phi\mid_{R_Q} = \Phi'\mid_{R_Q}$; in other words Φ and Φ' are equivalent if they determine the same element of $E_Q = \{R_Q \to M\}$. Consider a section of such foliation - a map $\tilde{\Phi} : E_Q \to \mathcal{R}_Q$ which assigns to each element of E_Q *a unique* element of $\mathcal{R}_Q$. In other words we set a rule which singles out one and only one solution to the problem (8), (9) for each $\Phi_0 = \Phi_{R_Q}$. Such section exists due to the stated assumptions about the space of solutions of the problem (8), (9). Consider now the following functional on E:

$$F[\Phi] = \int_\Sigma dV G_{ij}(\Phi)(\Phi^i - \tilde{\Phi}(\Phi_0)^i)(\Phi^j - \tilde{\Phi}(\Phi_0)^j), \tag{17}$$

Clearly , $\hat{Q}F[\Phi] = 0$. Using (17) we introduce the following deformation of the generating functional (13):

$$Z_\lambda[J] = \int [D\Phi]_E e^{i\beta(S[\Phi] + \int_\Sigma dV J_i \Phi^i + \lambda F[\Phi])} \tag{18}$$

Note that $Z[J] = Z_0[J]$. Let us show that $Z_\lambda[J]$ is in fact independent of λ:

$$\frac{\partial}{\partial\lambda} ln\ Z_\lambda[J] = i\beta \int_\Sigma dV < F[\Phi(p)] >_{\lambda,J}, \tag{19}$$

where $< \ >_{\lambda,J}$ denotes the average with respect to the "action" functional - an argument of exponent in (18). A correlator $< F[\Phi(p)] >_{\lambda,J}$ can be considered as a function on Σ. It follows from (17) that the restriction of this function to R_Q is zero. Moreover this function is Q-invariant as a consequence of the Q-symmetry of the problem. Really, $Q < F[\Phi] >_{\lambda,J} = < \hat{Q}F[\Phi] >_{\lambda,J} = 0$. The last equality can be regarded as a Ward identity corresponding to the Q-invariance of the vacuum of the theory at hand. Thus the r. h. s of (19) is an integral over Σ of a Q-invariant function equal to zero on zero locus R_Q of Q. Therefore it is equal to 0 in virtue of localization condition (1).

So, $Z_\lambda[J]$ is independent of λ. Thus one can compute the generating function $Z[J]$ as follows:

$$Z[J] = \lim_{\lambda\to\infty} Z_\lambda[J] \tag{20}$$

One can rewrite the r. h. s of (18) in the following form:

$$Z[J] = \lim_{\lambda\to\infty} \int [D\Phi_0]_{E_Q} \int_{\{\Phi_{R_Q} = \Phi_0\}} [D\Phi]_E e^{i\beta(S[\Phi] + \int_\Sigma dV J_i \Phi^i + \lambda F[\Phi])} \tag{21}$$

In the limit $\lambda \to \infty$ the internal integral in (21) localizes to the the critical points of the functional $S[\Phi] + \int_\Sigma dV + \int_\Sigma dV J_i \Phi^i + \lambda F[\Phi]$ which is defined on the space of maps having a fixed restriction to R_Q. It follows from the Q-invariance of this functional that one of these critical points if $\Phi = \tilde{\Phi}(\Phi_0)$ (see [6] for a proof in the even case). It can be shown under very general assumptions on $S[\Phi]$ that $\Phi = \tilde{\Phi}(\Phi_0)$ is the only extremum contributing to (21) in the limit $\lambda \to \infty$. The contribution can be calculated using infinite-dimensional version of the stationary phase method. As a result we obtain the following answer for the generating functional (14):

$$Z[J] = \int [D\Phi_0]_{E_Q} e^{-\beta S[\tilde{\Phi}(\Phi_0)] + \int_\Sigma dV J_i \tilde{\Phi}(\Phi_0)^i}, \tag{22}$$

where we absorbed the determinants which appeared as a result of computation of corresponding gaussian integrals into the redefinition of functional measure on E_Q. But now we note that in virtue of (10)

$$Q\mathcal{L}(\tilde{\Phi}(\Phi_0), \partial\tilde{\Phi}(\Phi_0)) = 0,$$

therefore the integral $S[\tilde{\Phi}(\Phi_0)] = \int_\Sigma \mathcal{L}(\tilde{\Phi}(\Phi_0), \partial\tilde{\Phi}(\Phi_0))$ localizes to the zero locus of the vector field Q. It follows also from the non-degeneracy of Q in the vicinity of R_Q that $\partial_\perp \tilde{\Phi}(\Phi_0) \mid_{R_Q} = 0$. This remark together with localization condition (1) permits us to conclude that

$$S[\tilde{\Phi}(\Phi_0)] = \int_{R_Q} dv_Q \mathcal{L}(\Phi_0, \partial'\Phi_0, 0) \tag{23}$$

The same localization arguments work for the source term as we have chosen the functions J's to be Q-invariant. Substituting (23) into (21) we arrive at the expression (15) for the generating functional of the reduced theory.

The way we established the equality between (14) and (15) is somewhat naive in the sense that the result was achieved by means of formal manipulations with the path integral without addressing the questions of proper renormalization of the loop expansion arriving. Our results only suggest the possibility of the phenomenon considered, an additional analysis is required in each particular case.

Keeping up with the level of generality adopted for the present section we can discuss the relation between instanton sectors in the original and the reduced theory. Suppose that Φ_0 is an extremum of the action functional $S_{red}[\Phi]$ of the reduced theory, $\Phi : R_Q \to M$. Let $\tilde{\Phi}(\Phi_0) \in \{\Sigma \to M\}$ be a Q-invariant map such that its restriction to $R_Q \subset \Sigma$ coincides with Φ_0. Then $\tilde{\Phi}$ is an extremum of the action functional $S[\Phi]$ of the original theory. The proof of this statement is based on the $\hat{Q}$-symmetry of $S[\Phi]$ and goes along the same line as its even counterpart (see [6]). Conversely, any Q-invariant extremum of the original theory produces a solution to the equations of motion of the

reduced theory by means of restriction. Moreover any two Q-invariant extrema $\tilde{\Phi}$ and $\tilde{\Phi}'$ of $S[\Phi]$ give rise to the same extremum of $S_{red}[\Phi]$ given that their restrictions to R_Q coincide, $\tilde{\Phi}_{R_Q} = \tilde{\Phi}'_{R_Q}$. Note also that $S[\tilde{\Phi}] = S[\tilde{\Phi}']$ in virtue of assumed localization of integrals over Σ with Q-invariant integrals. Thus we established a one-two-one correspondence between instantons of the reduced theory and critical submanifolds of E consisting of Q-invariant instantons of the original theory having a given restriction to Q. It follows from above that such BPS-like solutions completely determine the instantons contribution to the Q-invariant sector of the original theory. Really, if we suppose for example that the instantons Φ_q of the reduced theory are isolated and classified by an integer q, then by virtue of the equality (15) the instanton contribution to the partition function of the original (Wick rotated) theory is equal to

$$\sum_q \frac{e^{-\beta S_{red}[\Phi_q]}}{\sqrt{detHess(S_{red}[\Phi_q])}}, \tag{24}$$

and is clearly determined by Q-invariant extrema only.

Our conclusions concerning the dimensional reduction of supersymmetric field theories generalize and provide the geometrical understanding of the results of ref. [2]. The cited paper contains the first non-pertubative proof of the dimensional reduction of Parisi-Sourlas model and describes the relation between instanton sectors of Parisi-Sourlas model on a linear $(3,2)$ space and its reduction which is a bosonic theory in dimension 1.

3 Applications and conclusions.

In conclusion let us explain the relation of Parisi-Sourlas model to the discussion above.

Consider a supermanifold $\Sigma = B \times \mathcal{R}^{(2,2)}$, where B is a (super)manifold. Let M be a linear superspace. Let us choose local coordinates $\{x^i, y^\alpha, \theta, \overline{\theta}\}$ on Σ, where $\{x^i\}$ is a set of local coordinates on B, $\{y^\alpha, \theta, \overline{\theta}\}$ are even and odd coordinates on $\mathcal{R}^{(2,2)}$. Let h be a Riemannian metric on the manifold B. Then a metric on Σ can be defined by means of the following quadratic form:

$$g = h_{ij}(x)\delta x^i \delta x^j + \sum_\alpha \delta y^\alpha \delta y^\alpha + 2\delta\theta\delta\overline{\theta}, \tag{25}$$

Consider the following σ-model having Σ as a source manifold:

$$Z[\beta] = \int [d\Phi]_E e^{i\beta S[\Phi]}, \tag{26}$$

$$S[\Phi] = \int_\Sigma dV(g^{-1}(\Phi^I, \Phi^J) G_{I\ J}(\Phi) + V(\Phi)), \tag{27}$$

where $E = \{\Sigma \to M\}$, $\Phi \in E$; g^{-1} is a bivector field on Σ inverse to the quadratic form (25). In components $\Phi^I = \phi^I + \psi^I\overline{\theta} + \overline{\psi}^I\theta + A^I\theta\overline{\theta}$. It follows from the results of [4] that the model (26), (27) can be viewed as a result of stochastic quantization of a σ-model defined on the space of maps $\{B \times \mathcal{R}^{(2,0)} \to M\}$. Corresponding action functional is

$$S[\phi] = \int_{\Sigma_0} dV((h^{ij}\frac{\partial\phi^I}{\partial x^i}\frac{\partial\phi^J}{\partial x^j} + \sum_\alpha \frac{\partial\phi^I}{\partial y^\alpha}\frac{\partial\phi^J}{\partial y^\alpha})G_{I\ J}(\Phi) + V(\phi)), \tag{28}$$

In such interpretation we consider only the following correlation functions of the model (26), (27): $< \Phi^{I_1}\mid_B \ldots \Phi^{I_k}\mid_B >$. It is easy to check that the metric form (25) is invariant with respect to the following odd vector field on Σ:

$$Q = \overline{\theta}\frac{\partial}{\partial y^1} + \theta\frac{\partial}{\partial y^2} + y^2\frac{\partial}{\partial\overline{\theta}} - y^1\frac{\partial}{\partial\theta} \tag{29}$$

This vector field satisfies all conditions of the Corollary and the Theorem which yields the integration formula (1) for the integrals of Q-invariant functions over Σ.

Consider the Q-invariant sector of the model (26), (27). This sector describes in particular stochastic correlation functions of the model (28). The results of the previous section suggest that this sector is equivalent to the following model defined on the manifold B:

$$Z[\beta] = \int [d\phi]_{E_B} e^{iS[\phi]}, \tag{30}$$

$$S[\phi] = \int_B dv(g^{ij}\partial_i\phi^I\partial^j\phi^J G_{I\ J}(\phi) + V(\phi)), \tag{31}$$

where $E_B = \{B \to M\}$ and $\phi \in E_B$ and dv corresponds to the metric h on B. This conclusion agrees with corresponding statements about dimensional reduction of the original Parisi-Sourlas model and its modifications considered in [3].

Finally let us note that all results of the present section can be generalized to the case when Σ is a total space of a flat (m,m)-bundle over the base B.

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